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Using LQR Algorithm to Control Circular Two Stage Parallel Inverted Pendulum System

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ABSTRACT

This paper describes the dynamic models of a circular two-stage parallel inverted pendulum, which has been developed for the laboratory experiments. A circular two-stage parallel inverted pendulum has two different-length rigid pendulums which are connected to a horizontally rotating disc which is attached directly to a DC motor. The derivation of the dynamical equations and the linearized model are described. Finally, LQR algorithm is used to control the system.

Keywords: Circular Two-Stage Parallel Inverted Pendulum System, LQR Control, LQR Algorithm With Integral, Nonlinear Control

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1. INTRODUCTION

Inverted pendulum (Dynamic; Design) is one of widely used apparatus in control laboratories to demonstrate the modern control theory applications. The conventional inverted pendulum has the structure of the cart-type system, which has the limitation of the cart length. For this reason, the circular inverted pendulum was introduced to compensate this restriction.

Recently, a lot of researches on control of the inverted pendulum system (Real, 2009; Fuzzy; Yong, T.L., 1998; Futura and Yamakita, 1992) have been done. It has been known as a more difficult problem to derive full mathematical equations and design a controller for double inverted pendulum based on its strong nonlinearity and inherent instability.

The objective of this paper is to obtain the full derivation of the mathematical equations based on Lagrange equation, then using LQR algorithm (Adaptive; Model; Huynh Thai Hoang, 2006; Nguyen and Huynh, 2006; Nguyen Thi Phuong Ha, 2008) to make the system in equilibrium state with two pendulums upwards.

2. CIRCULAR TWO-STAGE PARALLEL INVERTED PENDULUM

The circular two-stage parallel inverted pendulum are described as the figure bellow (Fig. 1.)

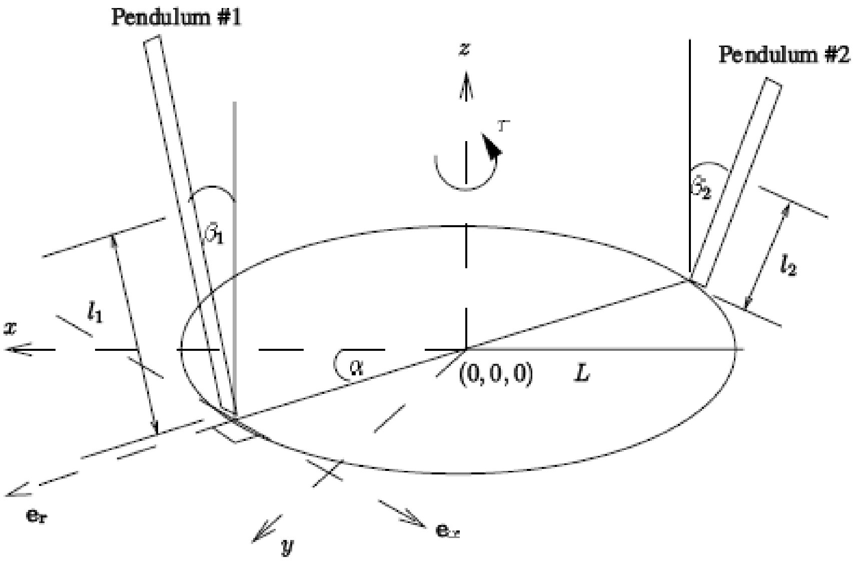

Figure 1. Circular two-stage parallel inverted pendulum

Two pendulums are connected to a rotating disc which is driven by a DC motor. The control objective is to balance two pendulums in the upright position. The system variables are described as follows.

τ: The external torque applied to the disc (N.m)

α: The angular displacement of the rotating disc (rad).

β_1: The 1st pendulum angle with respect to the vertical axis (rad).

β_2: The 2nd pendulum angle with respect to the vertical axis (rad).

The system parameters are described in Table 1.

Table 1. Parameters of rotary inverted pendulum system

Parameter	Notation	Unit
Inertia of the rotating disc	J_0	$kg.m^2$
Inertia of the 1st pendulum	J_1	$kg.m^2$
Inertia of the 2nd pendulum	J_2	$kg.m^2$
Viscous coef. of rotating disc	c_0	$Nm.s$
Viscous coef. of the 1st pendulum	c_1	$Nm.s$
Viscous coef. of the 2nd pendulum	c_2	$Nm.s$
Mass of the 1st pendulum	m_1	kg
Mass of the 2nd pendulum	m_2	kg
The length of the first pendulum	l_1	m
The length of the second pendulum	l_2	m
The radius of the rotating disc	L	m
The gravity constant	g	m/s^2
Torque constant	K_m	$N.m/A$
Back emf. constant	K_b	$Volt/rad$
Resistance in motor circuit	R	Ω

2.1. Nonlinear dynamic model

The mathematical equation of the system can be derived by using Lagrange equation $L = K - U$ where K is the kinetic energy, U is the potential energy and W is Rayleigh's dissipation function. The Lagrange equation is as follows.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial W}{\partial \dot{q}_i} = F_i \qquad (1)$$

Kinetic Energy

The total kinetic energy of the system is:

$$K = \frac{1}{2}J_0\dot{\alpha}^2 + \frac{1}{2}J_1\dot{\beta}_1^2 + \frac{1}{2}J_2\dot{\beta}_2^2 + \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \qquad (2)$$

Where v_1, v_2 respectively are the velocities of the center of mass of the 1st pendulum and the 2nd pendulum. After calculating, the equation of the kinetic energy is describe as follows.

$$K = \frac{1}{2}J_0\dot{\alpha}^2 + \frac{1}{2}J_1\dot{\beta}_1^2 + \frac{1}{2}J_2\dot{\beta}_2^2 + \frac{1}{2}m_1(l_1 sin\beta_1\dot{\alpha})^2 + \frac{1}{2}m_1(L\dot{\alpha})^2 + \frac{1}{2}m_1(l_1\dot{\beta}_1)^2 - m_1 l_1 L cos\beta_1\dot{\beta}_1\dot{\alpha} + \frac{1}{2}m_2(l_2 sin\beta_2\dot{\alpha})^2 + \frac{1}{2}m_2(L\dot{\alpha})^2 + \frac{1}{2}m_2(l_2\dot{\beta}_2)^2 - m_2 l_2 L cos\beta_2\dot{\beta}_2\dot{\alpha} \qquad (3)$$

Potential Energy

The equation of the system is:

$$U = m_1 g l_1 cos\beta_1 + m_2 g l_2 cos\beta_2 \qquad (4)$$

Loss Energy

The loss energy of the system depends on frictional force:

$$W = \frac{1}{2}c_0\dot{\alpha}^2 + \frac{1}{2}c_1\dot{\beta}_1^2 + \frac{1}{2}c_2\dot{\beta}_2^2 \tag{5}$$

From (1), (3), (4), (5) we can get the Lagrange equation as follows.

$$L = K - U = \frac{1}{2}J_0\dot{\alpha}^2 + \frac{1}{2}J_1\dot{\beta}_1^2 + \frac{1}{2}J_2\dot{\beta}_2^2 + \frac{1}{2}m_1(l_1 sin\beta_1\dot{\alpha})^2 + \frac{1}{2}m_1(L\dot{\alpha})^2 + \frac{1}{2}m_1(l_1\dot{\beta}_1)^2 - m_1l_1Lcos\beta_1\dot{\beta}_1\dot{\alpha} + \frac{1}{2}m_2(l_2 sin\beta_2\dot{\alpha})^2 + \frac{1}{2}m_2(L\dot{\alpha})^2 + \frac{1}{2}m_2(l_2\dot{\beta}_2)^2 - m_2l_2Lcos\beta_2\dot{\beta}_2\dot{\alpha} - m_1gl_1cos\beta_1 - m_2gl_2cos\beta_2 \tag{6}$$

The system is controlled by a DC servo motor. In case of neglecting the effect of inductor in the motor circuit, the relationship between moment τ and potential V is given by:

$$\tau = \frac{K_mV}{R} - \frac{K_mK_b\dot{\alpha}}{R} \tag{7}$$

From (6) and (7), we obtain the equation of the circular two-stage parallel inverted pendulum as:

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta}_1 \\ \ddot{\beta}_2 \end{bmatrix} + \begin{bmatrix} p'_1 \\ p'_2 \\ p'_3 \end{bmatrix} = \frac{K_m}{R} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

where

$$P_{11} = J_0 + m_1l_1^2sin^2\beta_1 + m_1L^2 + m_2l_2^2sin^2\beta_2 + m_2L^2 \tag{9}$$

$$P_{12} = -m_1l_1Lcos\beta_1 \tag{10}$$

$$P_{13} = -m_2l_2Lcos\beta_2 \tag{11}$$

$$P_{21} = -m_1l_1Lcos\beta_1 \tag{12}$$

$$P_{22} = J_1 + m_1l_1^2 \tag{13}$$

$$P_{23} = 0 \tag{14}$$

$$P_{31} = -m_2 l_2 L cos\beta_2 \tag{15}$$

$$P_{32} = 0 \tag{16}$$

$$P_{33} = J_2 + m_2 l_2^2 \tag{17}$$

and

$$p_1' = m_1 l_1^2 \dot{\beta}_1 \dot{\alpha} \sin(2\beta_1) + m_1 l_1 L \dot{\beta}_1^2 sin\beta_1 + c_0 \dot{\alpha} + m_2 l_2^2 \dot{\beta}_2 \dot{\alpha} \sin(2\beta_2) + \left(c_0 + \frac{K_m K_b}{R}\right) \dot{\alpha} \tag{18}$$

$$p_2' = -m_1 l_1^2 \dot{\alpha}^2 sin\beta_1 cos\beta_1 - m_1 g l_1 sin\beta_1 + c_1 \dot{\beta}_1 \tag{19}$$

$$p_2' = -m_2 l_2^2 \dot{\alpha}^2 sin\beta_2 cos\beta_2 - m_2 g l_2 sin\beta_2 + c_2 \dot{\beta}_2 \tag{20}$$

2.2. Linearized dynamic model

Because the control objective is keep two pendulum balance upright, so we can get the operation point: $\beta_1^* = \beta_2^* = 0$. And we know that, if x is small enough:

$$x^2 \approx 0, sinx \approx x, sin^2 x \approx 0$$

From then, let $x = [\alpha \quad \beta_1 \quad \beta_2 \quad \dot{\alpha} \quad \dot{\beta}_1 \quad \dot{\beta}_2]^T$ be a state variable, and $u = V$ be the input, we can linearize the equation (8) as below:

$$\dot{x} = Ax + Bu = E^{-1}Fx + E^{-1}Gu \tag{21}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_0 + m_1 L^2 + m_2 L^2 & -m_1 l_1 L & -m_2 l_2 L \\ 0 & 0 & 0 & -m_1 l_1 L & J_1 + m_1 l_1^2 & 0 \\ 0 & 0 & 0 & -m_2 l_2 L & 0 & J_2 + m_2 l_2^2 \end{bmatrix} \tag{22}$$

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\left(c_0 + \frac{K_m K_b}{R}\right) & 0 & 0 \\ 0 & m_1 g l_1 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & m_2 g l_2 & 0 & 0 & -c_2 \end{bmatrix} \quad (23)$$

$$G = \begin{bmatrix} 0 & 0 & 0 & \frac{K_m}{R} & 0 & 0 \end{bmatrix}^T \quad (24)$$

In the case of the *linear* system:

$$\dot{x} = Ax + Bu \quad (25)$$

or

$$x(k+1) = Ax(k) + bu(k) \quad (26)$$

and *quadratic* cost functions, the optimal control problem is said to be the *linear-quadratic* (LQ) optimal control problem. Further, for constant-coefficient matrices A and B and terminal time infinitely far in the future (meaning, of course, that the operating time is sufficiently long compared to the time constants of the system), the problem is referred to as the *infinite-horizon* or *infinite-time-to-go* problem. In this case, the control law simplifies to:

$$u = -Kx \quad (27)$$

with a constant-coefficient feedback gain matrix K. Matrix K can calculate from the LQR fuction:

K=lqr(A,B,Q,R,N)

or

Kd=dlqr(Ad,Bd,Q,R,N)

Difficulty in finding the right weighting factors limits the application of the LQR based controller synthesis.

3. USING THE LQR ALGORITHM TO COTROL THE CIRCULAR TWO-STAGE PARALLEL INVERTED PENDULUM SYSTEM

We use Simulink in Matlab to simulate the response of the system when using LQR algorithm. Here we assume that the inverted pendulum system is in balance with two pendulum uprights, so we can use the linearized equation (21) to find matrix K. We designed a nonlinear subsystem of a double inverted pendulum based on equation (8). In this simulation, the system is affected by control noise and output noises. The objective of the controller is to keep the system in balance when affected by external forces or noises. The simulation diagram is as follows (Fig. 2.)

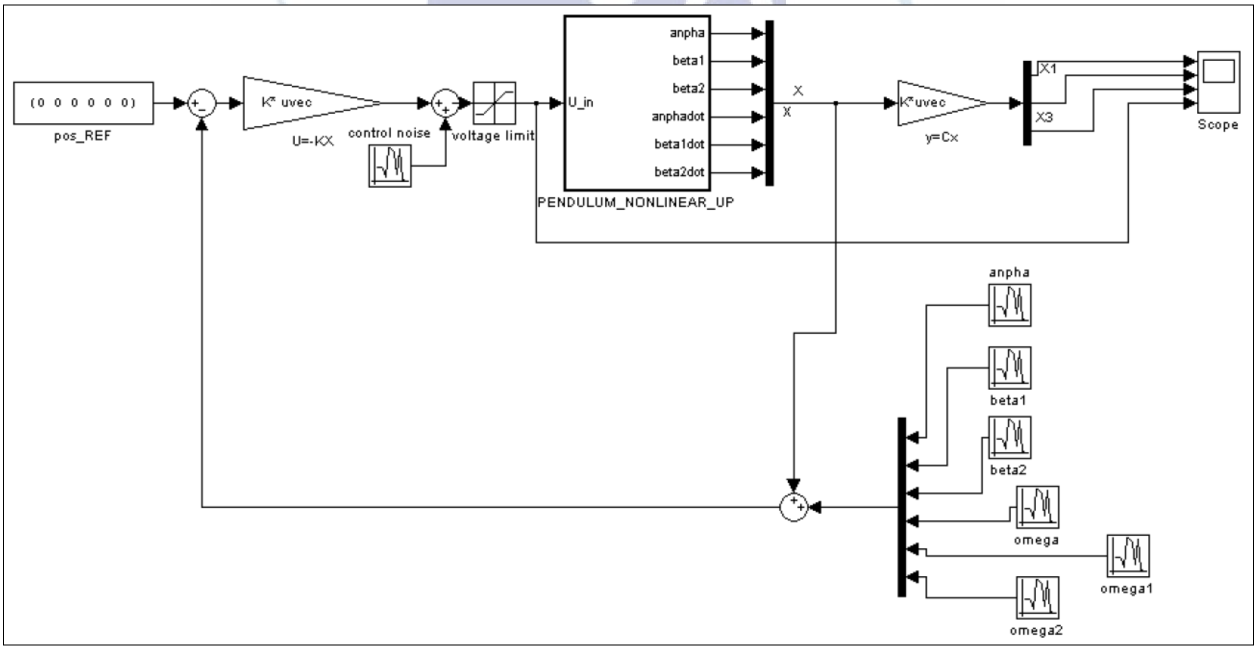

Figure 2. Simulink model for simulation

Here we choose the weighting matrix as below:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R = 0.04 \qquad (28)$$

Simulation results

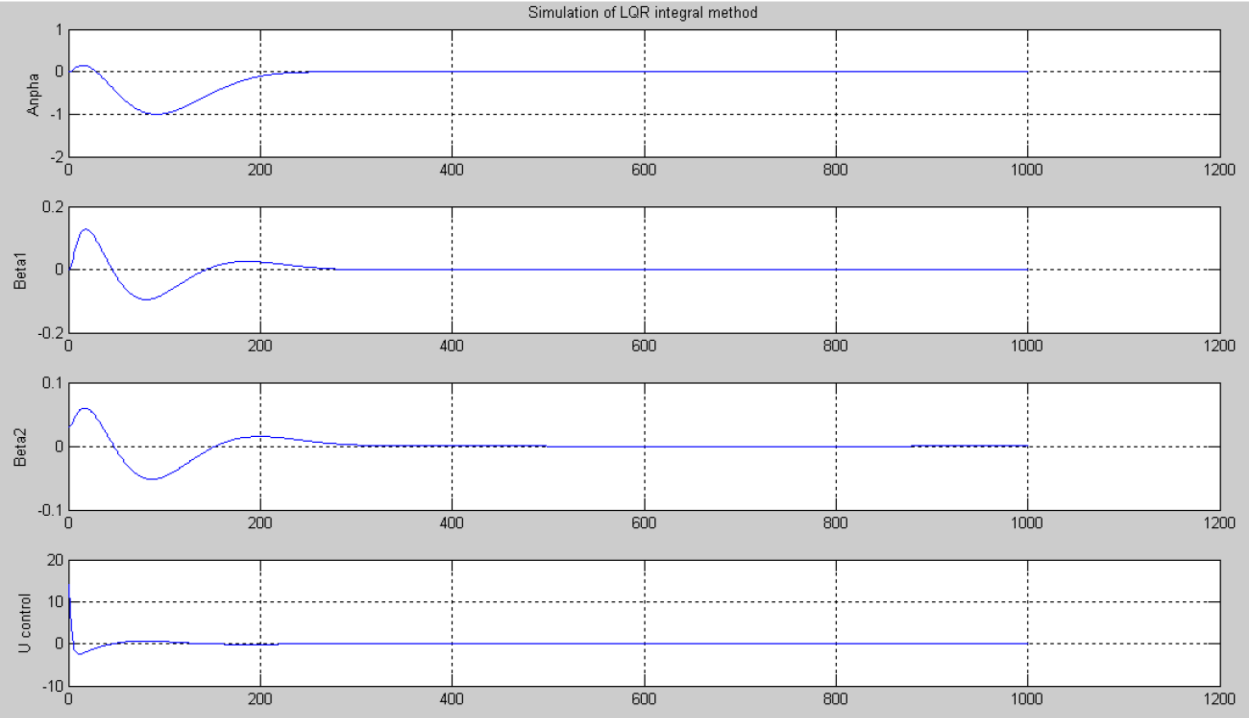

Figure 3. The response of the system without noise

Case 2: The response of the system when having noise (Fig. 4.)

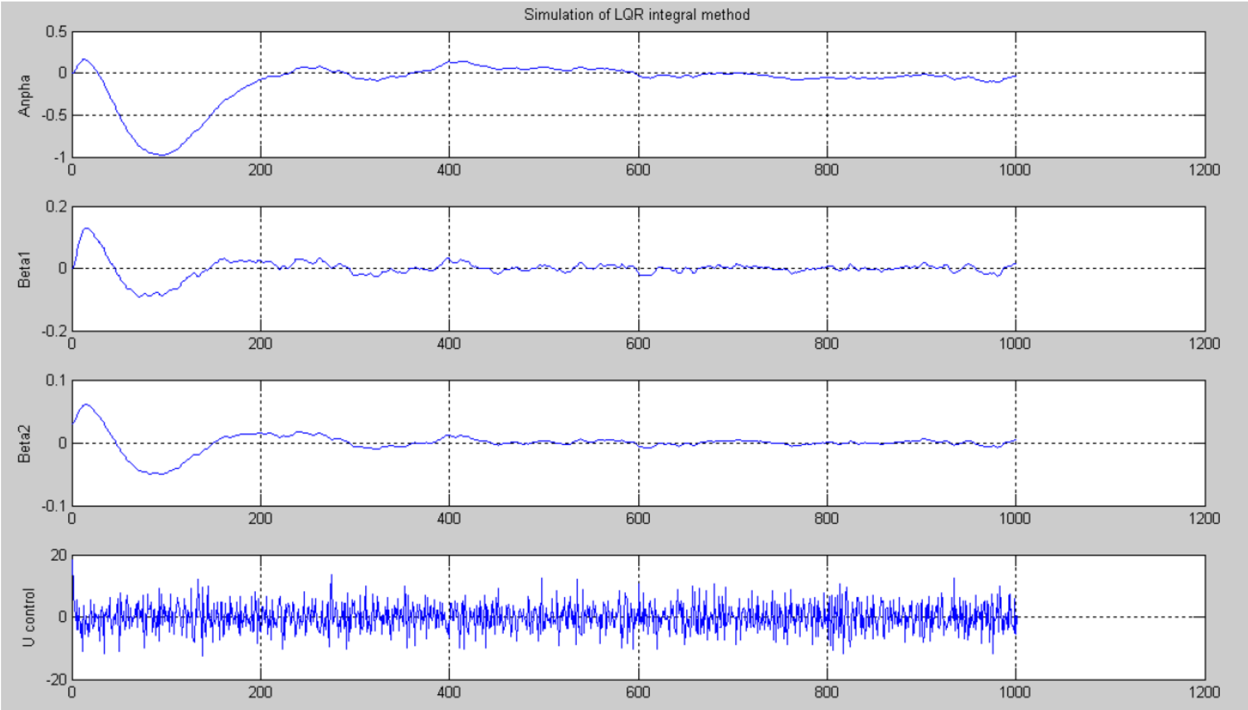

Figure 4. The response of the system with white noises, variance=1e-4

We can see that, the system has quite good responses even if having noise affected. Depending on methods of choosing the weighting matrix Q and R, the response of the system is different.

LQR Algorithm with Integral

To ensure that steady state errors are equal to zero, an integral is added in LQR controller. In this case, we have the extended state space equation with seven variables:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \ddot{\alpha} \\ \ddot{\beta}_1 \\ \ddot{\beta}_2 \\ \dot{xi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1.0027 & 0.5247 & -0.4650 & -0.0128 & -0.0067 \\ 0 & 35.2219 & 0.3837 & -0.3400 & -0.4487 & -0.0049 \\ 0 & 0.1720 & 8.1814 & -0.0798 & -0.0022 & -0.1051 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta_1 \\ \beta_2 \\ \dot{\alpha} \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3.8446 \\ 2.8114 \\ 0.6596 \\ 0 \end{bmatrix} u \qquad (29)$$

The simulation of LQR algorithm with integral is as follow:

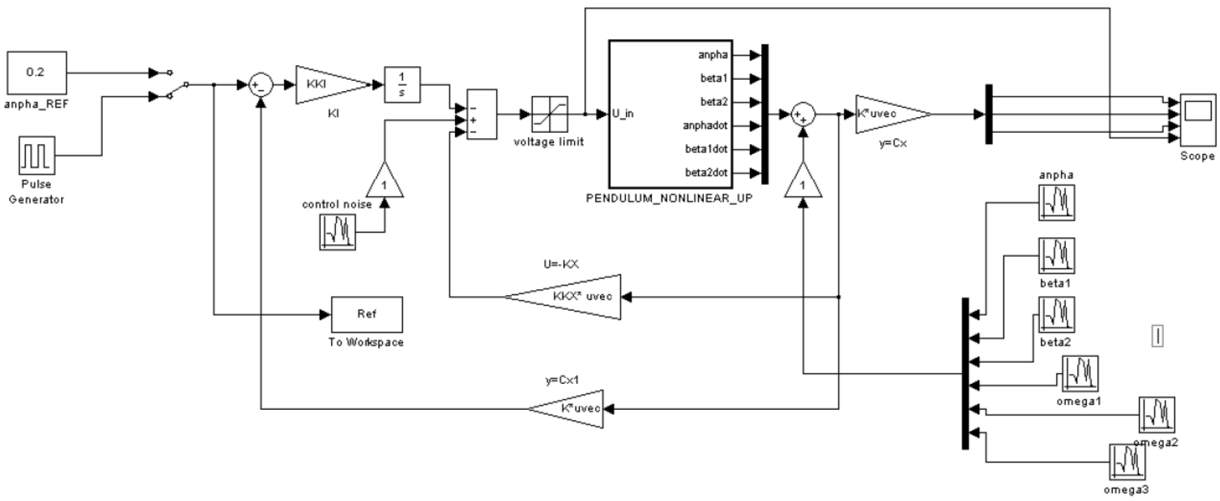

Figure 5. LQR controller with Integral

The weighting matrix is chosen as below:

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}, R = 0.4 \qquad (30)$$

Simulation results

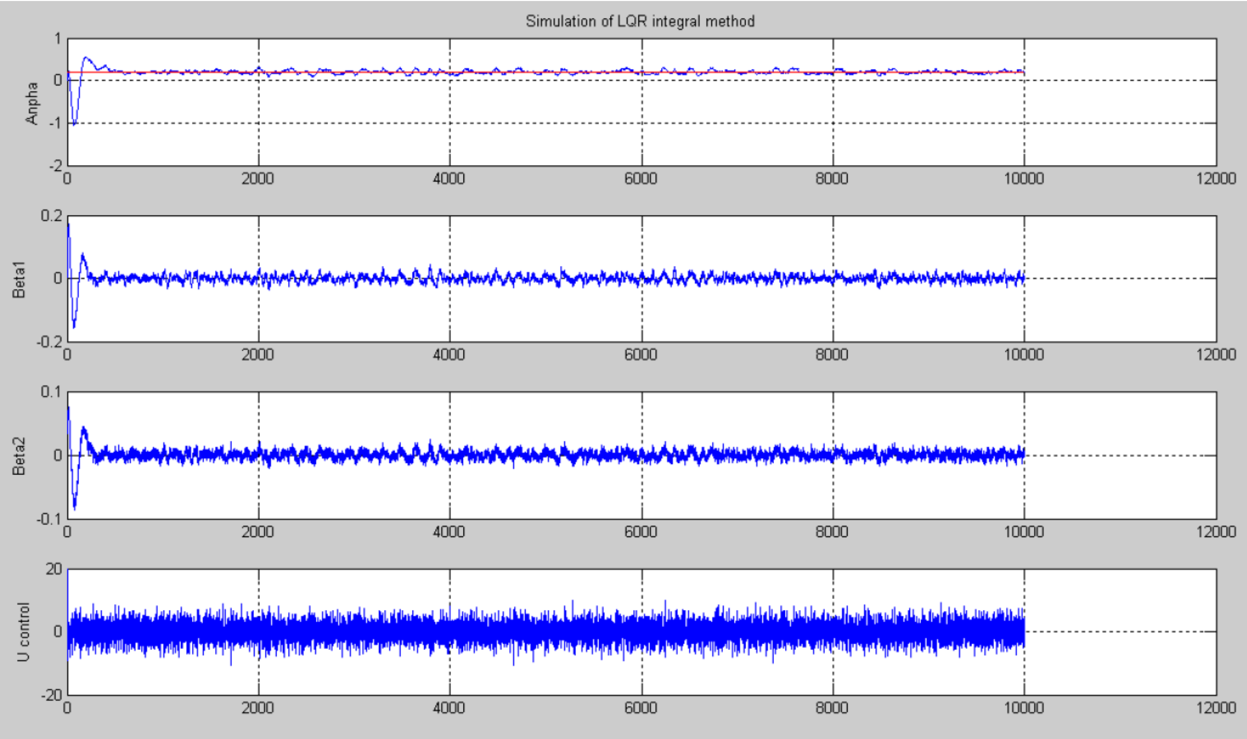

Figure 6. The response of the system with step signal, variance=1e-4

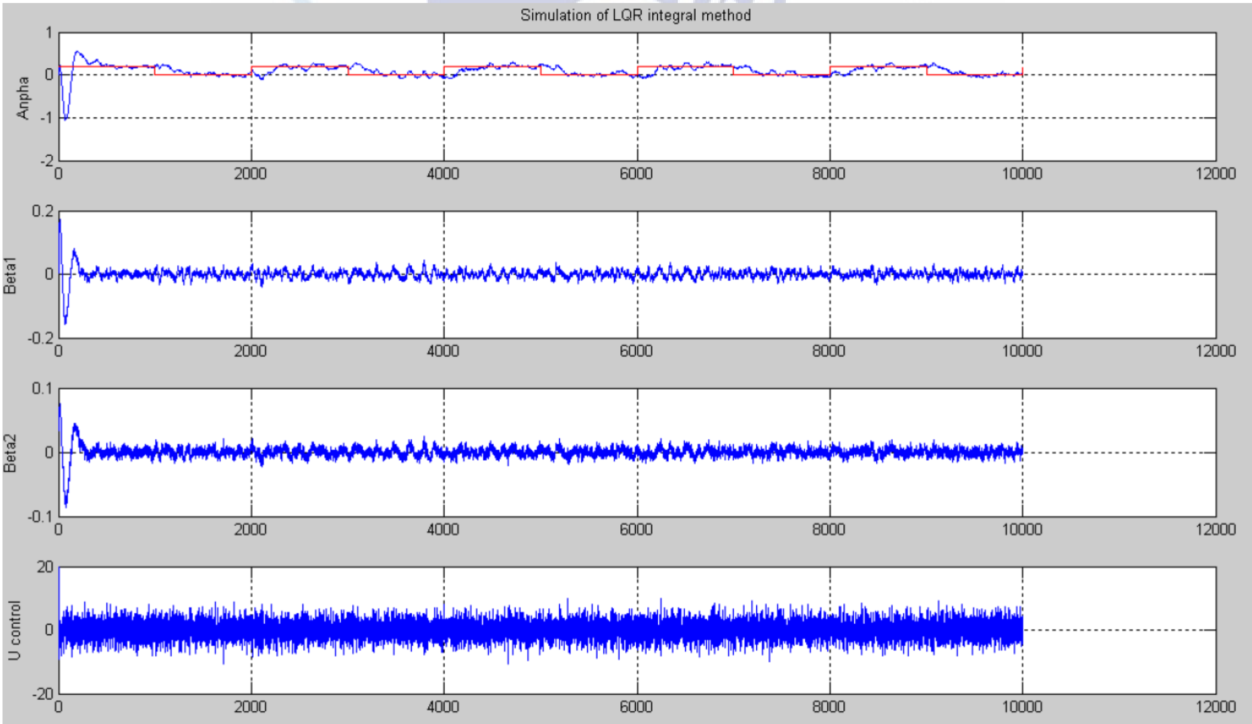

Figure 7. The response of the system with pulse signal, variance=1e-4

As we can see in Fig. 6 and Fig. 7, the response of the system can keep track of the step input signal and even the pulse input signal.

4. CONCLUSION

The dynamical equations of the circular two-stage parallel inverted pendulum system were derived in this work. Thereafter, the linearized equations were obtained at the specified operating points. Then, the LQR algorithm was used to control the system in balance.

REFERENCES

Adaptive Fuzzy Control Experiments And Comparative Analyses.

Dynamic models of a rotary double inverted pendulum system–Peerapat, Chulalongkong University.

Design, build and control a single/double inverted pendulum, University of Adelaide.

Fuzzy Saturating Control.

Futura, K., Yamakita, M., 1992, "Swing up control of inverted pendulum using pseudo-state feedback," *Proc Instn Mech Engrs*, Vol. 206.

Huynh Thai Hoang, 2006, "Intelligent Control Theory," *National University Publisher*.

Model free control and intelligent PID controllers.

Nguyen, T.P.H., Huynh, T.H., 2006, "Automatic Control Theory," *National University Publisher*.

Nguyen, T.P.H., 2008, "Modern Control Theory," *National University Publisher*.

Real-Time Optimal Control for Rotary Inverted Pendulum, *American Journal of Applied Sciences*, Vol. 6(6), pp. 1106-1115, 2009.

Yong, T.L., 1998, "State Space Predictive Control of Inverted Pendulum."