人文及管理學報　第四期
2007年11月　頁079~090
宜蘭大學人文及管理學院

A Note on Cross Section Integration

Chien-an Lung

Associate Professor, Humanities and Science Education Center,
National Ilan University

Abstract

The cross section integral $\int_0^h A(S_z)\,dz$ is studied for finding the volume of a solid, whose cross sections are bounded by Jordan (simple closed) curves $(x, y) = \gamma(t)$ instead of functions $y = f(x)$, $x = g(y)$. Although the method is elementary, it gives rise to some interesting problems.

Key Words: cross section, Jordan measurable set, content zero,

Jordan curve

截面積分的注記

龍健安

國立宜蘭大學人文暨科學教育中心副教授

摘　要

一立體之截面為喬登(簡單閉)曲線 $(x, y) = \gamma(t)$ 所圍，研究以截面積分公式 $\int_0^h A(S_z)dz$ 求立體體積。雖然使用基本方法，但因截面非由單變數函數 $y = f(x)$, $x = g(y)$ 所圍，而引發一些有趣問題。

關鍵詞：截面、喬登可測集、零容量、喬登曲線

1. Introduction

Consider the theorem (See [1]): "Let S be a bounded Jordan measurable 3-dimensional set lying between two planes $z=0$ and $z=h$. For each $z \in [0,h]$, let Sz be the cross section of S in the plane taken perpendicular to z-axis at z. Suppose for each $z \in [0,h]$, Sz is also Jordan measurable and let $A(S_Z)$ be the area of S_Z, then the Riemann integral $\int_0^h A(S_z)dz$ exists, and the volume $V(S)$ of S is given by $\int_0^h A(S_z)dz$". In most textbooks on advanced calculus, the proof involves the multiple integrals and the solid S is of the form $\{(x,y,z) | a \le x \le b, \varphi(x) \le y \le \psi(x), \Phi(x,y) \le z \le \Psi(x,y)\}$. The purpose of this paper is to prove the formula without the above assumption on S, but instead, with the cross sections of the solid being the regions bounded by simple closed curves. Thus, it is not surprising that some extra conditions will be added in order to prove it.

2. Curves

Recall that a subset S of R^n has (n-dimensional) content zero if for every $\varepsilon>0$ there is a finite cover $\{U_1,\cdots U_n\}$ of S consisting of closed rectangles such that $\sum_{i=1}^{n} v(U_i) < \varepsilon$, where $v(U)$ is the n-dimensional volume of U which is defined as $(b_1-a_1)\cdots(b_n-a_n)$ if U= $\{(x_1,\cdots x_n) | a_1 \le x_1 \le b_1,\cdots,a_n \le x_n \le b_n\}$

We list some properties of curves:

Theorem 1 (See [2])

(Jordan Curve Theorem) Let γ be a simple closed curve in the plane R^2, then $R^2 - \gamma$ has exactly two connected components whose common boundary is γ .

Lemma 1 (See [3])

Let $\gamma(t) = (x(t), y(t))$ be a plane curve, $t \in [a,b]$. If $x(t)$ or $y(t)$ has a bounded derivative on $[a,b]$, then γ has 2-dimensional content zero.

Lemma 2(See [4])

Let $\gamma(t) = (x(t), y(t))$ be a rectifiable plane curve $t \in [a,b]$. Then γ has 2-dimensional content zero.

3. The Solid

To describe a solid, we need the following hypothesis:

Hypothesis 1

Let $F(z,t)$ be a continuous mapping from $[0,h] \times [0,1]$ into R^2 such that $\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial z}$ are continuous on $[0,h] \times [0,1]$, and let $\gamma_z(t) := (x_z(t), y_z(t)) = F(z,t)$ be a simple closed C^2 curve for every $z \in [0,h], \gamma_z(0) = \gamma_z(1)$.

S is a solid lying between two planes $z = 0$ and $z = h$ such that P(Sz), the projection of Sz onto x-y plane , is just the region bounded by γ_z, for every $z \in [0,h]$.

Theorem 2

The mapping $z \to A(S_z)$ is continuous on $[0,h]$.
We divide the proof into 3 steps:

Step 1:

Since the mapping $[0,h] \times [0,1] \to R^2$ given by $(z,t) \to \gamma_z(t)$ is continuous, for any number $\varepsilon > 0$, then we can always find a number $\delta > 0$ such that for any $z_1, z_2 \in [0,h]$ and $t_1, t_2 \in [0,1]$, $|(z_1,t_1) - (z_2,t_2)| < \delta$ implies $|\gamma_{z_1}(t_1) - \gamma_{z_2}(t_2)| < \varepsilon$

Define the length of γ_z as $\Lambda(\gamma_z)=\sup \sum_{i=1}^{n} |\gamma_z(t_i) - \gamma_z(t_{i-1})|$ where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_n = 1$.

Step 2:

Let $\overline{B}((x,y),\varepsilon)$ be a closed disk of radius ε and center (x,y). For a fixed z_0, $\bigcup_{(x,y)\in\gamma_{z_0}} \overline{B}((x,y),\varepsilon)$ is Jordan measurable and $A(\bigcup_{(x,y)\in\gamma_{z_0}} \overline{B}((x,y),\varepsilon))=2\varepsilon\,\Lambda(\gamma_{z_0})$. (1)

Proof of (1):

(i) Since $\gamma_{z_0}(t)$ is a C^2 curve, $\Lambda(\gamma_{z_0}) < \infty$. Let $\lambda_{z_0}(s)$ be the reparametrization of γ by arc length. We claim that

$$\bigcup_{(x,y)\in\gamma_{z_0}} \overline{B}((x,y),\varepsilon) = \left\{ \lambda_{z_0}(s) + t\varepsilon(y'(s),-x'(s)) \,\middle|\, 0 \le s < \Lambda(\gamma_{z_0}), -1 \le t \le 1 \right\}. \quad (2)$$

Indeed, the $\supseteq$ part of (2) is obvious. For the $\subseteq$ part, first pick a point P in $\bigcup_{(x,y)\in\gamma_{z_0}} \overline{B}((x,y),\varepsilon)$. It follows from the compactness of γ_{z_0} and continuity of distance that we can find a point Q in γ_{z_0} such that the distance between P and Q is a minimum. Furthermore, the line $\overline{PQ}$ is perpendicular to γ_{z_0}, which can be seen by using $(v \bullet v)' = 2(v \bullet v')$. This completes the $\subseteq$ part of (2).

(ii) The area of the set

$\{\lambda_{z_0}(s) + t\varepsilon(y'(s), -x'(s)) \mid 0 \le s < \Lambda(\gamma_{z_0}), -1 \le t \le 1\}$ is just the difference of the areas of the two regions bounded by the two simple closed C^1 curves $s \to \lambda_{z_0}(s) \pm \varepsilon(y'(s), -x'(s))$ if ε is small enough. Hence, by Green's theorem,

$$A(\bigcup_{(x,y)\in\gamma_{z_0}} \overline{B}((x,y),\varepsilon)) =$$

$$\left| \int_0^{\Lambda(\gamma_{z_0})} (x + \varepsilon y')(y' - \varepsilon x'')ds - \int_0^{\Lambda(\gamma_{z_0})} (x - \varepsilon y')(y' + \varepsilon x'')ds \right| =$$

$$\left| -2\varepsilon \int_0^{\Lambda(\gamma_{z_0})} (x(s)x''(s) - (y'(s))^2)ds \right| =$$

$$\left| -2\varepsilon \int_0^{\Lambda(\gamma_{z_0})} (x(s)x''(s) + (x'(s))^2 - 1)ds \right| = \left| -2\varepsilon \int_0^{\Lambda(\gamma_{z_0})} ((x(s)x'(s))' - 1)ds \right|$$

$$= 2\varepsilon\, \Lambda(\gamma_{z_0}).$$

This proves (1).

Step 3:

Fix z_0, and let $\varepsilon > 0$. By continuity of the mapping $(z,t) \to \gamma_z(t)$, we can find a $\delta > 0$ such that $|\gamma_z(t) - \gamma_{z_0}(t)| < \varepsilon$ whenever $|z - z_0| < \delta$. Thus, for such z, the set $\{\gamma_z(t) \mid 0 \le t < 1\}$ is contained in $\bigcup_{(x,y)\in\gamma_{z_0}} \overline{B}((x,y),\varepsilon)$, which is the union of all closed disks of radius ε centered at (x,y) over the curve γ_{z_0}.

Since $A(\bigcup_{(x,y)\in\gamma_{z_0}} \overline{B}((x,y),\varepsilon)) = 2\varepsilon\Lambda(\gamma_{z_0})$, it follows that

$|A(S_z) - A(S_{z_0})| < 2\varepsilon\Lambda(\gamma_{z_0})$ whenever $|z - z_0| < \delta$.

This completes the proof of Theorem 2.

Theorem 3

If a solid S satisfies hypothesis 1 and its cross sections are increasing in the sense that $P(S_a) \subseteq P(S_b)$ whenever $0 \le a < b \le h$, then the volume $V(S)$ of S is given by $\int_0^h A(S_z)dz$.

Proof: Let $0 = z_0 < z_1 < z_2 < \cdots < z_n = h$ be a regular partition of $[0,h]$. It follows that $\sum_{i=1}^{n} A(S_{z_{i-1}})\Delta z_i \le V(S) \le \sum_{i=1}^{n} A(S_{z_i})\Delta z_i$. Letting $n \to \infty$, we are done.

Theorem 4

If a solid S satisfies hypothesis 1, then $V(S) = \int_0^h A(S_z)dz$

Proof: Step 1:

Since $[0,h]\times[0,1]$ is compact, F is uniformly continuous. For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that for every $a, b \in [0,h]$ and $t_1, t_2 \in [0,1]$, we have $|(a,t_1)-(b,t_2)| < \delta$ implying $|\gamma_a(t_1) - \gamma_b(t_2)| < \varepsilon$. Let $0 = z_0 < z_1 < z_2 < \cdots < z_n = h$ be a partition of $[0,h]$ with norm $< \delta$. Then $\gamma_z \subseteq \bigcup_{(x,y)\in\gamma_{z_i}} \overline{B}((x,y),\varepsilon)$ for any $z \in [z_{i-1}, z_i]$. Define $M_i = \bigcup_{z\in[z_{i-1},z_i]} P(S_z)$, $m_i = \bigcap_{z\in[z_{i-1},z_i]} P(S_z)$. Then $m_i \subseteq P(S_z) \subseteq M_i$ for any $z \in [z_{i-1}, z_i]$, thus $\underline{A}(m_i) \le A(S_z) \le \overline{A}(M_i)$ where $\overline{A}$ and $\underline{A}$ are Jordan outer and inner areas respectively.

Step 2:

Since $\dfrac{\partial F}{\partial t}$ is continuous on $[0,h]\times[0,1]$, the mapping $z \to \int_0^1 |\gamma_z'(t)|dt$

is continuous on $[0,h]$. It follows that $\sup_{z\in[0,h]} \Lambda(\gamma_z) < \infty$. Let $\sup_{z\in[0,h]} \Lambda(\gamma_z)$:=L. From the fact that $\gamma_z \subseteq \bigcup_{(x,y)\in\gamma_{z_i}} \overline{B}((x,y),\varepsilon)$ for every $z \in [z_{i-1}, z_i]$, and $\bigcup_{(x,y)\in\gamma_{z_i}} \overline{B}((x,y),\varepsilon)$ is contained in the region between two non-intersecting simple closed curves, we obtain that $\overline{A}(M_i) - \underline{A}(m_i) < 2\varepsilon\Lambda(\gamma_{z_i}) < 2\varepsilon L$.

Since $\sum_{i=1}^{n} \underline{A}(m_i)\Delta z_i \le V(S) \le \sum_{i=1}^{n} \overline{A}(M_i)\Delta z_i$, and

$\sum_{i=1}^{n} \underline{A}(m_i)\Delta z_i \le \sum_{i=1}^{n} A(S_{z_i})\Delta z_i \le \sum_{i=1}^{n} \overline{A}(M_i)\Delta z_i$, It follows that

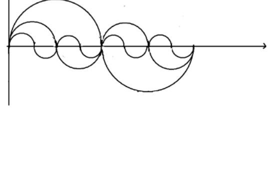
$\left|(\sum_{i=1}^{n} A(S_{z_i})\Delta z_i) - V(S)\right| < 2\varepsilon hL$. This completes the proof of Theorem 4.

Remarks:

(1) The solid is Jordan measurable since its boundary is a union of a C^1 mapping image of $[0,h] \times [0,1]$ and two plane regions bounded by C^2 curves.

(2) Let $F(z,t)$ be a continuous mapping from $[0,h] \times [0,1]$ into R^2. For every $z \in [0,h]$, $\gamma_z(t) := (x_z(t), y_z(t)) = F(z,t)$ is a simple closed piecewise C1 curve.

(i) In general, the mapping $[0,h] \to R$ given by $z \to \Lambda(\gamma_z)$ may not be continuous as can seen in the diagram below: a sequence of functions with a constant arc-length converges pointwise to a function whose length is smaller.



(ii) Although γ_z is rectifiable for every $z \in [0,h]$, the set $\{\Lambda(\gamma_z) | z \in [0,h]\}$ may still be unbounded. Indeed, for example, if $\gamma_{\frac{1}{n}}(t)$ and $\gamma_0(t)$ are simple closed curves such that $\gamma_{\frac{1}{n}}(t)$=(t, $\frac{\sin 2\pi nt}{\sqrt{n}}$) and $\gamma_0(t)$=(t, 0) for $t \in [0, \frac{1}{2}]$ then $\Lambda(\gamma_{\frac{1}{n}}) > 2\sqrt{n} \to \infty$ as $n \to \infty$.

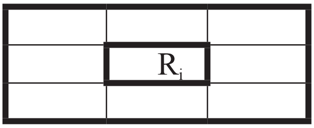
4. Some Interesting Problems

(1) Let $\overline{B}(x,\delta) = \{y \in R^2 \| x - y | \le \delta\}$. If a set S in R^2 has content zero, then

$$\lim_{\delta \to o}(\overline{A}(\bigcup_{x\in S}\overline{B}(x,\delta)) = 0.$$

Proof: Let ε >0 be given and S be contained in a union of n rectangles R_1, R_2,--,R_n such that $\sum_{i=1}^{n} A(R_i) < \frac{\varepsilon}{9}$ .If m is the minimum of the 2n side lengths of the n rectangles and $\delta < m$, then we claim that $\overline{A}(\bigcup_{x\in S}\overline{B}(x,\delta) < \varepsilon$. Indeed, let S_i=S$\bigcap R_i$ and pick any point x_i in S_i, then $\overline{B}(x_i,\delta)$ is contained in a rectangle $\mathbf{R_i}$ whose side lengths are three times of those of R_i (See diagram below). Hence, $\overline{B}(x_i,\delta) \subseteq \mathbf{R_i}$. Since the choice of x_i in S_i is arbitrary, we have $\bigcup_{x\in S_i}\overline{B}(x,\delta) \subseteq \mathbf{R_i}$ and $\bigcup_{x\in S}\overline{B}(x,\delta) \subseteq \bigcup_{i=1}^{n}\mathbf{R_i}$, which implies

$$\overline{A}(\bigcup_{x\in S}\overline{B}(x,\delta) < A(\bigcup_{i=1}^{n}\mathbf{R_i}) < \sum_{i=1}^{n} A(\mathbf{R_i}) = 9\sum_{i=1}^{n} A(R_i) < \varepsilon$$



(2) Assume that S is any bounded set in R^2 and δ >0. Is $\bigcup_{x\in S}\overline{B}(x,\delta)$ a Jordan measurable set?

Remark: If δ is replace by δ_x, the value of which depends on x, then $\bigcup_{x\in S}\overline{B}(x,\delta_x)$ is not always a Jordan measurable set. For example, let S=$\{x_1,x_2,x_3,\cdots\}$ be the rational points in the interior of the unit square $[0,1]^2$, then for each x_i in S, we can find a disk $\overline{B}(x_i,\delta_{x_i})$ contained in the interior of $[0,1]^2$ such that $\sum_{i=1}^{\infty}A(\overline{B}(x_i,\delta_{x_i}))<1$, Since the boundary of $\bigcup_{x\in S}\overline{B}(x,\delta_x)$ is $[0,1]^2-$interior of $\bigcup_{x\in S}\overline{B}(x,\delta_x)$. It follows that the measure of the boundary of $\bigcup_{x\in S}\overline{B}(x,\delta_x)$ is positive, which deduces that $\bigcup_{x\in S}\overline{B}(x,\delta_x)$ is not a Jordan measurable set.

(3) Let $\gamma_3:[0,1]\to[0,1]^2$ be a closed curve (See diagram 1) such that $\gamma_3(t)$ passes through square i for $t\in[\frac{i-1}{4^3},\frac{i}{4^3}]$, and $\gamma_4:[0,1]\to[0,1]^2$ (See diagram 2) be a closed curve such that $\gamma_4(t)$ passes through square i for $t\in[\frac{i-1}{4^4},\frac{i}{4^4}]$. Let $\gamma_n(t)$ be defined similarly. It follows that $\gamma_n(t)\to\gamma(t)$ as $n\to\infty$, but $\gamma(t)$ is not a simple closed curve ($\gamma(0)=\gamma(\frac{1}{2})=\gamma(1)$, $\gamma(\frac{7}{4^k})=\gamma(\frac{9}{4^k})$, k=2,3,4,---), and the image of γ is $[0,1]^2$.

22	21	20	17	16	13	12	11
23	24	19	18	15	14	9	10
26	25	30	31	2	3	8	7
27	28	29	32	1	4	5	6
38	37	36	33	64	61	60	59
39	40	35	34	63	62	57	58
42	41	46	47	50	51	56	55
43	44	45	48	49	52	53	54

(Diagram 1)

86	85	84	81	80	79	66	65	64	63	50	49	48	45	44	43
87	88	83	82	77	78	67	68	61	62	51	52	47	46	41	42
90	89	94	95	76	73	72	69	60	57	56	53	34	35	40	39
91	92	93	96	75	74	71	70	59	58	55	54	33	36	37	38
102	101	100	97	118	119	122	123	6	7	10	11	32	29	28	27
103	104	99	98	117	120	121	124	5	8	9	12	31	30	25	26
106	105	110	111	116	115	126	125	4	3	14	13	18	19	24	23
107	108	109	112	113	114	127	128	1	2	15	16	17	20	21	22
150	149	148	145	144	143	130	129	256	255	242	241	240	237	236	235
151	152	147	146	141	142	131	132	253	254	243	244	239	238	233	234
154	153	158	159	140	137	136	133	252	249	248	245	226	227	232	231
155	156	157	160	139	138	135	134	251	250	247	246	225	228	229	230
166	165	164	161	182	183	186	187	198	199	202	203	224	221	220	219
167	168	163	162	181	184	185	188	197	200	201	204	223	222	217	218
170	169	174	175	180	179	190	189	196	195	206	205	210	211	216	215
171	172	173	176	177	178	191	192	193	194	207	208	209	212	213	214

(Diagram 2)

(4) Is it true that a simple closed curve in R^2 has content zero? The answer is no. See [5].

References

[1] Spivak, Michal. *calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*, A. Benjamin, Inc., New York-Amsterdam, 1965.

[2] Chuan-chi Hsiung. *A First Course in Differential Geometry*, John Wiley & Sons, 1981.

[3] Flatto, Leopold. *Advanced Calculus*, Williams and Wilkins,1976.

[4] Apostol, T. *Mathematical Analysis*, 2nd ed., Addison Wesley, 1974.

[5] WF Osgood. ‘A Jordan curve of positive area’s, *Transactions of the American Mathematical Society* (1903).

[6] Rudin, W. *Principles of Mathematical Analysis*, McGraw-Hill, 1964.