Part I 選擇題 (30%，共六題，每題5分，答錯倒扣1分，作答時請在答案卷寫上題號後直接填入答案)

1. If the solution to the initial-value problem $\begin{cases} y'' - 4y' + 13y = 0 \\ y(0) = 4, \\ y(0) = -1. \end{cases}$ is presented as

$y(x) = e^{cx}(\alpha \cos ax + \beta \sin bx)$ and both a and b are positive, then which is incorrect? (A) $a = b$ (B) $a = 2$ (C) $\alpha = 4$ (D) $\beta = -3$ (E) $c = 2$.

2. Among the following equations, which one is not exact differential?
   (A) $(y^2 - 1)dx + (2xy - \sin y)dy = 0$
   (B) $y^2 dx - (2 + 3y^2 - 2xy)dy = 0$
   (C) $(1 + \ln xy)dx + (1 + x / y^2)dy = 0$
   (D) $(2x + 3y - 2)dx + (3x - 4y + 6)dy = 0$
   (E) $(3x^2 + y\cos x)dx + (\sin x - 4y^3)dy = 0$.

3. The integral $\int_{(0,0)}^{(1,1)} \left[(6xy^2 - y^3)dx + (6x^2 y - 3xy^2)dy\right]$ = (A) 0 (B) 2 (C) -2 (D) 22 (E) -22.

4. Which of the following sets of vectors is linear dependent?
   (A) $(1,0)$, $(0,1)$
   (B) $(1,2)^T$, $(3,4)^T$
   (C) $(1,2,3)$, $(4,5,6)$, $(7,8,9)$
   (D) $p_1(x) = x^2 + x - 1$, $p_2(x) = x^2 - x + 1$, $p_1(x) = -x^2 + x + 1$
   (E) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 5 & 8 \\ 6 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

5. Which one of the following is not an equivalent statement for an $n \times n$ Nonsingular Matrix $\mathbf{A}$? (A) rows of $\mathbf{A}$ are linear dependent (B) $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every right side $\mathbf{b}$ (C) $\det(\mathbf{A}) \neq 0$ (D) $Rank(\mathbf{A}) = n$ (E) $\mathbf{A}$ has an inverse.

6. Given that $\mathbf{A} = \begin{bmatrix} 9 & -1 \\ 2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -3 \\ 2 & 3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -1 & 0 & -2 \\ 3 & -2 & 0 \\ 4 & 2 & 7 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -5 & 7 & 3 & 0 \\ 11 & 4 & -9 & 4 \end{bmatrix}$, and $\mathbf{E} = \mathbf{AB}$,

which one owns the largest determinant? (A) **A** (B) **B** (C) **C** (D) **D** (E) **E**.

Part II 計算題 (70%，共五題，每題14分)

1. Let $Dy = \dfrac{dy}{dx}$ and $D^2 y = \dfrac{d^2 y}{dx^2}$. Solve the differential equation $(D^2 + 2D + 4)y = 4x^2 + 6e^{-x} - \sin x$.

2. Let $\mathbf{A} = \begin{bmatrix} 5 & -8 & -1 \\ 4 & -7 & -4 \\ 0 & 0 & 4 \end{bmatrix}$. (i) Find the eigenvalues and eigenvectors of $\mathbf{A}$. (ii) Find a complete solution of the system $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

3. Let $f(t)$ is a periodic function whose definition in one period is

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases}, \quad 0 \le t < 2\pi .$$

(i) Find the Laplace transform of $f(t)$, namely $F(s) = \int_0^\infty e^{-st} f(t) dt$.

(ii) Find the Fourier series of $f(t)$ such that $f(t) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \dfrac{n\pi t}{L} + b_n \sin \dfrac{n\pi t}{L} \right)$, $L = \pi$.

(iii) Let the Fourier series be reformulated as $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \cdot n \cdot t}$. What is the relationships of c_n to a_n and b_n?

4. $B = \{(1,0),(0,1)\}$ and $B' = \{(1,-1),(-1,2)\}$ are two bases of R^2, and $T(x, y) = (3x, x - y)$ is a linear operator on R^2. (i) Find $\mathbf{P}$, the transition matrix from B' to B. (ii) Determine the transformation matrix of T with respect to B. (iii) Use a similarity transformation to find the matrix of T with respect to B'. (iv) Suppose the coordinate vector of $\mathbf{u}$ with respect to B is $\mathbf{a} = (3,4)$. Find the coordinate vector with respect to B'. (v) Let the solutions of Parts (ii), (iii), (iv) be $\mathbf{A}$, $\mathbf{A}'$, and $\mathbf{a}'$, respectively. What is the relationship between $\mathbf{Aa}$ and $\mathbf{A}'\mathbf{a}'$?

5. $\mathbf{v}_1 = (1,-2,2)$ and $\mathbf{v}_2 = (0,3,-6)$ are linearly independent in R^3. These two vectors form a basis for a two-dimensional subspace W of R^3. (i) Use the Gram-Schmidt process to construct an orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2\}$ from $\{\mathbf{v}_1, \mathbf{v}_2\}$. (ii) Given that $\mathbf{x} = (9,0,9)$, decompose $\mathbf{x}$ into the sum of a vector that lies in W and a vector orthogonal to W.