Discrete Mathematics and Linear Algebra

1. **(10%)** Show that among n+1 arbitrarily chosen integers, there are two whose difference is divisible by n.

2. **(10%)** The edges of K_6 (a complete graph on 6 vertices) are to be painted either red or blue. Show that for any arbitrary way of painting the edges there is a red K_3 (a K_3 with all its edges painted red) or a blue K_3 .

3. **(10%)** Determine the sum

$$\binom{n}{1}+2\binom{n}{2}+\cdots+n\binom{n}{n} .$$

4. Let a_r denote the number of edges in a complete graph on r vertices.
   (a) **(10%)** Derive a recurrence relation for a_r in terms of a_{r-1}.
   (b) **(10%)** Solve the recurrence relation.

5. **(10%)** Let $T : R^2 \to R^2$ be defined by T(a,b) = $(|a|,b)$. Show that T is not linear.

6. **(10%)** Find the value of c which makes it possible to solve the system

$$\begin{cases} x+\ \ y+2z=2 \\ 2x+3y-\ \ z=5 \\ 3x+4y+\ \ z=c. \end{cases}$$

7. **(10%)** Find an orthonormal basis for the subspace spanned by (1,-1,0,0), (0,1,-1,0), and (0,0,1,-1).

8. Let

$$M=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

   (a) **(10%)** Find the rank and the eigenvalues of M.
   (b) **(10%)** Show that M is diagonalizable.