Question 1

Determine whether each of the following “theorems” is TRUE or FALSE.
Assume that a, b, c, d, and m are integers with $m > 1$.

(1) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv b + d \pmod{m}$. [2 marks]

(2) If $a \equiv b \pmod{m}$, then $2a \equiv 2b \pmod{2m}$. [2 marks]

(3) If $a \equiv b \pmod{m^2}$, then $a \equiv b \pmod{m}$. [2 marks]

(4) If $a \ \mathbf{mod}\ m = b \ \mathbf{mod}\ m$, then $a \equiv b \pmod{m}$. [2 marks]

(5) If $ac \equiv bc \pmod{m}$, then $a \equiv b \pmod{m}$. [2 marks]

Question 2

(1) In the questions below suppose $A = \{x, y\}$ and $B = \{x, \{x\}\}$. Determine whether each of the following statements is TRUE or FALSE.

(i) $x \subseteq B$. [2 marks]

(ii) $\varnothing \in P(B)$. [2 marks]

(iii) $\{x\} \subseteq A - B$. [2 marks]

(iv) $\varnothing \subseteq A$. [2 marks]

(v) $\{x, y\} \in A \times A$. [2 marks]

(2) Suppose the universal set is {1, 2, 3, 4, 5, 6, 7, 8} and
A = {1, 2, 3, 6, 8}, B = {6, 7, 8}, C = {3, 6}.

(i) Write the set $(\overline{A \cup C}) \cup B$ in enumerated form. [3 marks]

(ii) Write the set $C^2 \cap (A \times B)$ in enumerated form. [3 marks]

Question 3

(1) For primitive statements p and q, determine if the compound statement

$$(p \vee q) \rightarrow [q \rightarrow (p \wedge q)]$$

is a tautology by constructing a truth table for it. [6 marks]

(2) For primitive statements p, q, r, and s, use the Laws of Logic table (at page 5) to simplify the compound statement below. Please provide the steps and reasons. [6 marks]

$$[[[(p \wedge q) \wedge r] \vee [(p \wedge q) \wedge \neg r]] \vee \neg q] \rightarrow s.$$

(3) Determine whether the following argument is valid. Use the Rules of Inference table (at page 5) to justify your answer, and please provide the steps and reasons. [6 marks]

If the band could not play rock music or the cake were not delivered on time, then Petty's birthday party would have been canceled and Petty would have been angry. If the birthday party were canceled, then refunds would have had to be made. No refunds were made. Therefore the band could play rock music.

Question 4

In each of the questions below determine whether the binary relation P is:
reflexive (R), symmetric (S), antisymmetric (A), and/or transitive (T).

(1) The relation P on $\{1, 2, 3, \ldots\}$ where aPb means $a \mid b$. [2 marks]

(2) The relation P on the set of all integers where aPb means $a^2 = b^2$. [2 marks]

(3) The relation P on the set of all people where aPb means that a is younger than b. [2 marks]

(4) The relation P on the set of all integers where aPb means $a \neq b$. [2 marks]

(5) The relation P on the set of all integers where aPb means $|a - b| \leq 1$. [2 marks]

Question 5

(1) How many edges are there in the complete bipartite graph $\boldsymbol{K}_{3,7}$? [2 marks]

(2) How many 0s are there in the adjacency matrix for $\boldsymbol{K}_{3,7}$? [2 marks]

(3) What is the length of the longest path in graph $\boldsymbol{K}_{3,7}$? [2 marks]

(4) What is the chromatic number of graph $\boldsymbol{K}_{3,7}$? [2 marks]

(5) How many edges are there in the complete graph $\boldsymbol{K}_7$? [2 marks]

(6) What is the largest value of n for which $\boldsymbol{K}_n$ is planar? [2 marks]

(7) What is the length of the longest simple cycle in the wheel graph $\boldsymbol{W}_{10}$? [2 marks]

(8) What is the minimum height of a binary tree with 100 vertices? [2 marks]

Question 6

Consider the following recursive algorithm, named *exp*, which takes a real number x and a natural number y as inputs and is supposed to output the value x^y.

Algorithm *exp*(x, y)
  if $y = 0$ then return (1)
  else
    $z := exp(x, \left\lfloor \frac{y}{2} \right\rfloor)$
    if y is even then return ($z \times z$)
    else return ($x \times z \times z$)

(1) Hand-turn the algorithm to compute exp(5, 5). Please show all your workings.
[4 marks]

(2) Prove by induction on y that the algorithm is correct. [2 marks]

Question 7

For the calculations below **please show all your workings**. You may leave your answer in the form of an arithmetic expression (sums, products, ratios, factorials).

(1) How many arrangements are there of the 12 letters in SOCIOLOGICAL? [5 marks]

(2) In how many of the arrangements in part (1) are all the vowels adjacent? [2 marks]

(3) Find the coefficient of x^5y^2 in the expansion of $(2x – 3y)^7$? [4 marks]

(4) How many ways can 3 men and 3 women be seated about a round table in such a manner that the sexes alternate? [3 marks]

Question 8

For the calculations below **please show all your workings**. You may leave your answer in the form of an arithmetic expression (sums, products, ratios, factorials).

(1) What is the probability that the sum of the numbers on two dice is even when they are rolled? [3 marks]

(2) What is the probability that a fair coin lands Heads 4 times out of 5 flips? [3 marks]

(3) There are ten girls and ten boys in the class. If five of the 20 students are selected at random to form a discussion group, what is the probability that the discussion group consists of five students of the same sex? [4 marks]

Law(s)		Name
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$		equivalence law
$p \rightarrow q \equiv \neg p \vee q$		implication law
$\neg\neg p \equiv p$		double negation law
$p \wedge p \equiv p$	$p \vee p \equiv p$	idempotent laws
$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$	commutative laws
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	associative laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	de Morgan's laws
$p \wedge \mathrm{T} \equiv p$	$p \vee \mathrm{F} \equiv p$	identity laws
$p \wedge \mathrm{F} \equiv \mathrm{F}$	$p \vee \mathrm{T} \equiv \mathrm{T}$	annihilation laws
$p \wedge \neg p \equiv \mathrm{F}$	$p \vee \neg p \equiv \mathrm{T}$	inverse laws
$p \wedge (p \vee q) \equiv p$	$p \vee (p \wedge q) \equiv p$	absorption laws

Rule of Inference	Related Logical Implication	Name of Rule
1) p; $p \rightarrow q$; $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Rule of Detachment (Modus Ponens)
2) $p \rightarrow q$; $q \rightarrow r$; $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Law of the Syllogism
3) $p \rightarrow q$; $\neg q$; $\therefore \neg p$	$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$	Modus Tollens
4) p; q; $\therefore p \wedge q$		Rule of Conjunction
5) $p \vee q$; $\neg p$; $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Rule of Disjunctive Syllogism
6) $\neg p \rightarrow F_0$; $\therefore p$	$(\neg p \rightarrow F_0) \rightarrow p$	Rule of Contradiction
7) $p \wedge q$; $\therefore p$	$(p \wedge q) \rightarrow p$	Rule of Conjunctive Simplification
8) p; $\therefore p \vee q$	$p \rightarrow p \vee q$	Rule of Disjunctive Amplification
9) $p \wedge q$; $p \rightarrow (q \rightarrow r)$; $\therefore r$	$[(p \wedge q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow r$	Rule of Conditional Proof
10) $p \rightarrow r$; $q \rightarrow r$; $\therefore (p \vee q) \rightarrow r$	$[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$	Rule for Proof by Cases
11) $p \rightarrow q$; $r \rightarrow s$; $p \vee r$; $\therefore q \vee s$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$	Rule of the Constructive Dilemma
12) $p \rightarrow q$; $r \rightarrow s$; $\neg q \vee \neg s$; $\therefore \neg p \vee \neg r$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$	Rule of the Destructive Dilemma